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LETTER TO THE EDITOR

An exactly soluble two-dimensional Ising model with magnetic field

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Abstract. An Ising model formulated on a Kagome lattice with anisotropic ferromagnetic and antiferromagnetic interactions, and a magnetic field, is found to be exactly solvable for arbitrary values of temperature. The magnetic field acts on two of the three sublattices of the Kagome lattice. Explicit expressions for the partition function and the critical variety of the model are given.

The two-dimensional Ising model with a magnetic field is a long-standing unsolved problem of statistical mechanics. The only exact solutions that present critical behaviour for real magnetic fields are the hard-hexagon (Baxter 1980) and the super-exchange (Fisher 1960) models. Despite the fact that the latter model is decorated, its critical behaviour describes well that of antiferromagnets in a field (Fisher 1960, Kaufman 1987). However, there is no frustration in this system and no exact results are known for frustrated magnets in a field. It is then important to have exact results for Ising models in a field defined on a regular lattice and which contain frustration. This has motivated the present work.

In this letter an Ising model in a magnetic field formulated on the Kagome lattice (KL) is solved exactly for all temperatures and fields. The KL is one of the four regular lattices that tilt the plane in such a way that all sites and bonds are equivalent. Examples of interesting realisations of this network include Frank–Kasper layered crystalline alloys (Sachdev and Nelson 1985) and also iron atoms in jarosite-type materials (Townsend *et al* 1986). The model under study is shown in figure 1(a). We distinguish three interpenetrating sublattices, denoted by A, B and C, in such a way that no two sites of the same type are adjacent. Interaction J_1 (respectively $-J_1$) is taken between sites belonging to sublattices A and B (respectively C), while spins of sublattices B and C interact through interaction J_2. The magnetic field h acts only on sublattices B and C: spins of A type have zero magnetic moment. Otherwise stated, the spins of B and C type interact, in the vertical direction, through a mediating non-magnetic spin, leading to a model for superexchange on the Kagome lattice. Note that one recovers the decorated model of Fisher (1960) in the limit $J_2=0$.

The Boltzmann weight associated with the elementary cell (figure 1(b)) of the KL is as follows:

$$W(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5) = \exp[K_1\sigma_5(\sigma_2+\sigma_4-\sigma_1-\sigma_3)+K_2(\sigma_1\sigma_2+\sigma_3\sigma_4)+\tfrac{1}{2}H(\sigma_1+\sigma_2+\sigma_3+\sigma_4)] \tag{1}$$

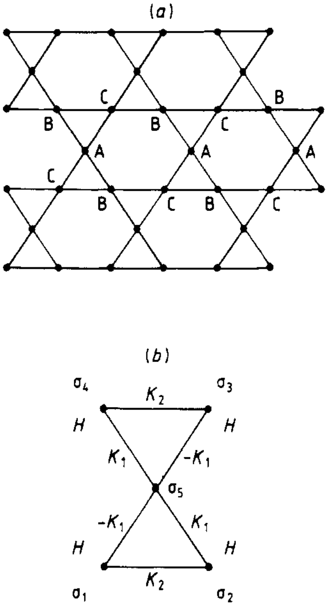

Figure 1. (*a*) A part of the Kagome lattice which is divided into three interpenetrating sublattices A, B and C. (*b*) An elementary cell of the Kagome lattice showing the Ising spins σ_i associated with each site. The two interaction parameters K_1 and K_2 and the reduced magnetic field H are also shown.

where $K_i = J_i/kT$ $(i = 1, 2)$, $H = mh/kT$, T being the temperature, k the Boltzmann constant and m the magnetic moment.

The values of J_1, J_2 and h are arbitrary and one can distinguish two distinct regions of the parameter space. When $h < 2J_1 - 2J_2$ the ground state (GS) is antiferromagnetic (where the J_1 and $-J_1$ bonds are satisfied) and two-fold degenerated. For $h > 2J_1 - 2J_2$ the GS is paramagnetic with all spins belonging to sublattices B and C parallel to the applied magnetic field, leaving the spins of A type free to flip. When $h = 2J_1 - 2J_2$ the GS is infinitely degenerated, with zero point entropy.

Notice that, for $J_2 > 0$, the system is frustrated while when $J_2 < 0$ it is unfrustrated. It is surprising that the GS structure does not depend on frustration. Moreover, as is well known, in zero field the antiferromagnetic Kagome lattice Ising model possesses high degeneracy of its GS and as can be seen, when a magnetic field is applied, this degeneracy is ruled out except, of course, when $h = 2J_1 - 2J_2$.

The partition function of the model is given by

$$Z_{\text{Kag}}(K_1, K_2, H) = \sum_{\{s\}} \pi_{\text{cells}}(W) \tag{2}$$

where the sum is performed over all spin configurations and the product is taken over all elementary cells of a lattice with $3N$ sites and periodic boundary conditions. Note that the partition function is invariant under the reverse of K_1 and H, independently. In order to obtain the exact solution of this model, we employed the same procedure used by one of the present authors in deriving an exact result for the KL Ising model with a magnetic field acting on all the spins of the lattice (Giacomini 1988). First, the KL is decorated by introducing an Ising spin s_i at the centre of each elementary triangle of the lattice as indicated in figure 2. The Boltzmann weight (1) can now be expressed

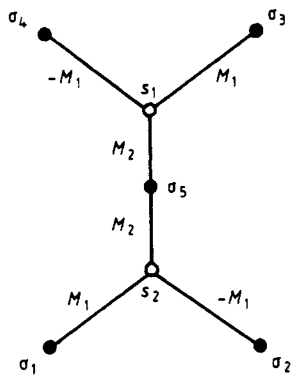

Figure 2. The decorated Kagome lattice. Full circles indicate the Ising spins σ_i of the original lattice and open circles represent the decorating Ising spins s_1 and s_2. The interactions M_1 and M_2 between both types of spins are also shown.

as follows:

$$W(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = R^2 \sum_{(s1,s2)} \exp[s_1(M_1\sigma_3 - M_1\sigma_4 + M_2\sigma_5) + s_2(M_1\sigma_1 - M_1\sigma_2 + M_2\sigma_5) + \tfrac{1}{2}H(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)]. \tag{3}$$

The parameters M_i and R are related to the K_i by the well known star-triangle relations (see, for example, Syozi 1972)

$$\cosh(2M_1) = \cosh(2K_1)\exp(-2K_2) \tag{4}$$

$$\cosh(2M_2) = 1 - (2\sinh^2(2K_1))/[\exp(4K_2) - 1] \tag{5}$$

$$R^2 = \tfrac{1}{2}(\cosh^2(2M_1) + \sinh^2(2M_2) + 2\cosh^2(2M_1)\cosh(2M_2))^{-1/2}. \tag{6}$$

The partition function of the KL model can now be expressed in terms of the partition function of the decorated model as follows:

$$Z_{\mathrm{Kag}}(K_1, K_2, H) = R^{2N} Z_{\mathrm{dec}}(M_1, M_2, H). \tag{7}$$

In this way the spins σ_i become decoupled and can be summed up. After the decimation of the spins σ_i, the resulting system is the Ising model on the honeycomb lattice with $2N$ sites, interactions L_1, L_2 and magnetic field H^*. Therefore the partition function of the KL model is expressed in terms of the partition function of the honeycomb lattice model (figure 3):

$$Z_{\mathrm{Kag}}(K_1, K_2, H) = (R^2A^2B)^N Z_{\mathrm{honey}}(L_1, L_2, H^*) \tag{8}$$

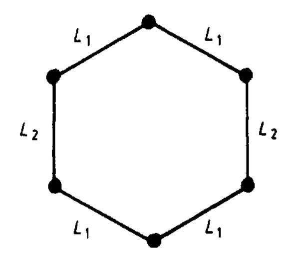

Figure 3. An elementary cell of the honeycomb lattice showing the interaction parameters L_1 and L_2.

where L_1, L_2, A, B and H^* are given by

$$\exp(4L_1) = \cosh(2M_1 + H)\cosh(2M_1 - H)(\cosh(H))^{-2} \tag{9a}$$

$$\exp(2L_2) = \cosh(2M_2) \tag{9b}$$

$$H^* = H_1 + H_2 \tag{10}$$

with

$$\exp(4H_1) = \cosh(2M_1 + H)(\cosh(2M_1 - H))^{-1} \tag{11}$$

$$\exp(4H_2) = \cosh(-2M_1 + H)(\cosh(2M_1 + H))^{-1} \tag{12}$$

and

$$A = 2(\cosh(2M_1 + H)\cosh(2M_1 - H)\cosh^2(H))^{1/4} \tag{13}$$

$$B = 2(\cosh(2M_2))^{1/2}. \tag{14}$$

A remarkable result is that, with the choice of the interaction parameters we made, the field H^* of the honeycomb lattice vanishes (see (10)-(12)) for arbitrary values of K_1, K_2 and H. Therefore, the KL model given by (1) is equivalent to the honeycomb lattice Ising model with interactions L_1 and L_2 and zero magnetic field ($H^* = 0$), which, as is well known, is exactly soluble.

On the other hand, when the magnetic field acts on all spins of the KL, there are three contributions to the field H^* of the honeycomb lattice, which can be made zero only when a *temperature*-dependent relation between the parameters of the KL is satisfied (Giacomini 1988).

Returning to the present model, and taking into account (6), (8), (13) and (14), the free energy per spin ψ in the thermodynamic limit is given by

$$\psi_{\text{Kag}}(K_1, K_2, H) = (-kT/3)\log(R^2A^2B) + \tfrac{2}{3}\psi_{\text{honey}}(L_1, L_2, H^* = 0). \tag{15}$$

the explicit expression for ψ_{honey} can be found, for example, in Syozi (1972).

Taking into account (4), (5) and (9), the interactions L_1 and L_2 of the honeycomb lattice are given in terms of K_1, K_2 and H, by the following expressions:

$$\exp(2L_2) = 1 - 2\sinh^2(2K_1)/[\exp(4K_2) - 1] \tag{16}$$

$$\exp(4L_1) = 1 + (\cosh^2(2K_1)\exp(-4K_2) - 1)(\cosh(H))^{-2}. \tag{17}$$

We are now able to discuss the critical behaviour of the model. As seen from (16) and (17), there are two different regions in the parameter space K_1 and K_2.

(i) If $K_2 < 0$, L_1 and L_2 are positive; the critical line of the corresponding honeycomb model is given by

$$\sinh(2L_1) = \operatorname{cotanh}(L_2). \tag{18}$$

(ii) If $K_2 < 0$, when $\exp(4K_2) > \cosh(4K_1)$, both L_1 and L_2 are real, with $L_1L_2 > 0$, and one has again the critical line (18); when $\exp(4K_2) < \cosh(4K_1)$, L_1 is real but L_2 is complex with $L_2 = L_2' + \mathrm{i}\pi/2$ (L_2' real) and $L_1L_2' > 0$. It can be shown that, in this case, the critical line of the corresponding honeycomb model is still given by (18) by taking $L_2 = L_2' + \mathrm{i}\pi/2$. Therefore, taking into account (16)-(18), the critical line of the Kagome lattice Ising model is now given, in terms of the original parameters K_1, K_2 and H, by the following expression:

$$\cosh^2(H) = \tfrac{1}{2}\{1 - \cosh^2(2K_1)\exp(-4K_2) + [(1 - \cosh^2(2K_1)\exp(-4K_2))^2 + \sinh^4(2K_1)\exp(-8K_2)]^{1/2}\}. \tag{19}$$

When K_2 is equal to zero one obtains the critical line of the Fisher model:

$$\cosh^2(H) = \sinh^2(2K_1)/(2+2\sqrt{2}). \tag{20}$$

Equation (19) always has a real solution for the critical field if

$$\cosh(2K_1) \geqslant \exp(2K_2) + [1+\exp(4K_2)]^{1/2}. \tag{21}$$

When the equality holds in (21), it gives the critical line for $H=0$, as can be easily deduced from (19). We show in figure 4 the critical lines for two representative regions of the parameters: $J_2 = \frac{1}{2}J_1$ (frustrated) and $J_2 = -\frac{1}{2}J_1$ (unfrustrated). Note the different order of magnitude for the critical field of both lines. This can be understood by remarking that, in the frustrated region, the ground state is nearer to the paramagnetic state than the non-frustrated one because the magnetic field, in our model, acts in the same sense as the interaction J_2 (when $J_2 > 0$). As a consequence one needs a smaller field to break the long-range order in the former case.

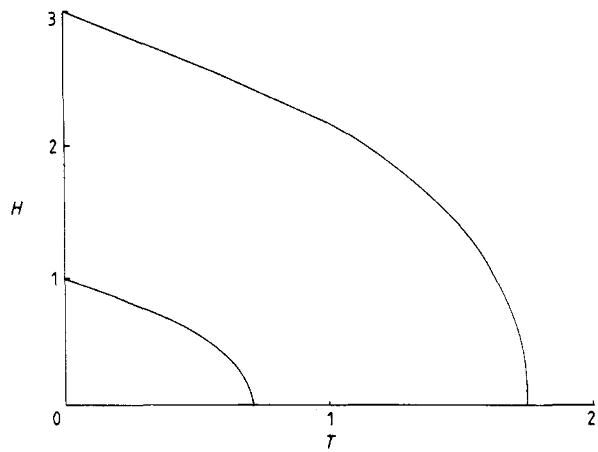

Figure 4. The critical magnetic field h as a function of the critical temperature T_c for the case $J_2 = \frac{1}{2}J_1$ (lower curve) and $J_2 = -\frac{1}{2}J_1$ (upper curve), respectively.

The model studied above possesses a disorder line with dimensional reduction when $\cosh(2K_1) = \exp(2K_2)$. In this case the Kagome model is equivalent to the honeycomb model with $L_1 = 0$ and becomes zero dimensional (see figure 3). It is interesting to note that this disorder line does not depend on the magnetic field.

Before concluding, let us emphasise that one can transform the honeycomb lattice Ising model with parameters L_1 and L_2 to another KL Ising model with new parameters K_1', K_2' and $H' = 0$. Therefore, the exact solubility of the present model is a consequence of the fact that one is able to define effective renormalised parameters that are, of course, functions of the original parameters on the KL and with a zero effective magnetic field. In conclusion, we have found a non-trivial exactly soluble Ising model with a magnetic field, defined on a regular lattice and which contains frustration. The complete statistical behaviour of this model is now under study and will be published elsewhere.

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References

Baxter R J 1980 *J. Phys. A: Math. Gen.* **13** L61
Fisher M E 1960 *Proc. R. Soc.* A **254** 66
Giacomini H J 1988 *J. Phys. A: Math. Gen.* **21** L31
Kaufman M 1987 *Phys. Rev.* B **36** 3697
Sachdev S and Nelson D R 1985 *Phys. Rev.* B **32** 1480
Syozi I 1972 *Phase Transitions and Critical Phenomena* vol 1, ed C Domb and M S Green (New York: Academic) p 270
Townsend M G, Longworth G and Rondant E 1986 *Phys. Rev.* B **33** 4919